Perth Modern School

PERTH MODERN SCHOOL
Exceptional schooling. Exceptional students.

Semester Two Examination, 2016

Question/Answer Booklet

MATHEMATICS SPECIALIST UNITS 3 AND 4

Section One: Calculator-free

Student Name: ____Solutions____________________

Teachers Name: ________________________

Student Number: In figures

In words ________________________________

Time allowed for this section

Reading time before commencing work: five minutes
Working time for section: fifty minutes

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer Booklet
Formula Sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight (8)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(14 marks)**

(a) Evaluate $\int_0^{\pi/4} \left(2+2\tan^2(x)\right) dx.$ (3)

(a) $\int_0^{\pi/4} 2+2\tan^2(x)\, dx \qquad 1+\tan^2(x)=\sec^2(x)$ ✓

$= 2\int_0^{\pi/4} \sec^2(x)\, dx$ ✓

$= 2\left[\tan(x)\right]_0^{\pi/4}$

$2(1-0)=2$ ✓

(b) Evaluate

$\int_0^1 e^{2x}\sqrt{1+e^{2x}}\, dx$

put $u=1+e^{2x}$

$\frac{du}{dx}=2e^{2x}$

$\frac{du}{2}=e^{2x}dx$ ✓

If $x=1, u=1+e^2$

If $x=0, u=2$ ✓

$=\int_2^{1+e^2} \sqrt{u}\,\frac{du}{2}$

$=\int_2^{1+e^2} \left(u^{1/2}\right) dx$

$=\left[\frac{2u^{3/2}}{3}\right]_2^{1+e^2}$ ✓

$=\frac{2}{3}\left(\sqrt{\left(1+e^2\right)^3}-\sqrt{2^3}\right)$ ✓

$=\frac{2}{3}\left(\sqrt{\left(1+e^2\right)^3}-2\sqrt{2}\right) = \frac{(1+e^2)^{3/2}}{3} - \frac{2\sqrt{2}}{3}$

(4)

(c) Determine $\int_e^{e^2} \frac{dx}{x\,ln(x)}$ using the substitution $u = ln(x)$ (3)

$$\int_e^{e^2} \frac{dx}{x\,ln(x)}$$

put $u = ln(x)$

$$\frac{du}{dx} = \frac{1}{x} \quad ✓$$

$$du = \frac{dx}{x}$$

If $x = e^2,\ u = ln(e^2) = 2\,lne = \underline{\underline{2}}$

✓ If $x = e,\ u = ln(e) = 1$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= [ln(u)]_1^2 \quad ✓$$

$$= ln(2) - ln(1)$$

$$\underline{= ln(2)}$$

(d) Determine $\int \frac{-3dx}{(x-2)(x+1)}$

HINT: Use partial fractions. (4)

$$\int \frac{-3dx}{(x-2)(x+1)} = -3\int \frac{dx}{(x-2)(x+1)}$$

$$\frac{1}{(x-2)(x+1)} = \frac{a}{(x-2)} + \frac{b}{(x+1)}$$

$$\frac{0x+1}{(x-2)(x+1)} = \frac{a(x+1)+b(x-2)}{(x-2)(x+1)} \quad ✓$$

$$= \frac{x(a+b)+(a-2b)}{(x-2)(x+1)} \quad ✓$$

Equating coefficients

$a + b = 0$ $\quad$ $a - 2b = \cancel{1}\ -3$

$a = -b$ $\quad$ $-3b = \cancel{1}\ -3$

$a = \cancel{\frac{1}{3}}\ -1$ $\quad$ $b = \cancel{-\frac{1}{3}}\ 1$

$$\int \frac{-3dx}{(x-2)(x+1)} = -3\left(\int\left(\frac{\cancel{\tfrac{1}{3}}\ -1}{(x-2)} + \frac{\cancel{\tfrac{1}{3}}\ 1}{(x+1)}\right)dx\right) \quad ✓$$

$$= \int\left(\frac{1}{(x+1)} - \frac{1}{(x-2)}\right)dx$$

$$\boxed{= ln(x+1) - ln(x-2) + c} \quad ✓$$

$$\int \frac{-3dx}{(x-2)(x+1)} = ln\frac{(x+1)}{(x-2)} + c \quad \text{OR}$$

Question 2 **(5 marks)**

Solve the complex equation $z^4 = -16$. (5)

$$z^4 = -16$$
$$= 2^4(-1)$$
$$= 2^4\left(cis\left(\pi + n(2\pi)\right)\right)$$
$$z = 2\left(cis\left(\pi + n(2\pi)\right)\right)^{\frac{1}{4}}$$
$$z = 2\left(cis\left(\frac{\pi}{4} + \frac{n\pi}{2}\right)\right)$$

$n = 0$ $\quad z = 2\left(cis\left(\frac{\pi}{4}\right)\right) = 2\left(cos\left(\frac{\pi}{4}\right) + i\,sin\left(\frac{\pi}{4}\right)\right) = \sqrt{2}(1+i)$

$n = 1$ $\quad z = 2\left(cis\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right) = 2\left(cos\left(\frac{3\pi}{4}\right) + i\,sin\left(\frac{3\pi}{4}\right)\right) = \sqrt{2}(-1+i)$

$n = 2$ $\quad \cancel{z = 2\left(cis\left(\frac{5\pi}{4}\right)\right)}$

$n = -1$ $\quad z = 2\left(cis\left(-\frac{\pi}{4}\right)\right) = \sqrt{2}(1-i)$

$n = -2$ $\quad z = 2\left(cis\left(-\frac{3\pi}{4}\right)\right) = \sqrt{2}(-1-i)$

$z = \sqrt{2}(1+i),\ \sqrt{2}(-1+i),\ \sqrt{2}(1-i),\ \sqrt{2}(-1-i)$

✓ ✓ ✓ ✓

$\frac{\pi}{4}, \frac{3\pi}{4}, \frac{-\pi}{4}, -\frac{3\pi}{4}$

Don't remove marks as **Not** asked for Cartesian Form

Question 3 **(5 marks)**

(a) Sketch $\{z : |x-1+iy| = 2|x+i(y-1)|\}$ on the set of axes below. (3)

$$|x-1+iy| = 2|x+i(y-1)|$$

$$\sqrt{(x-1)^2+y^2} = 2\sqrt{x^2+(y-1)^2}$$

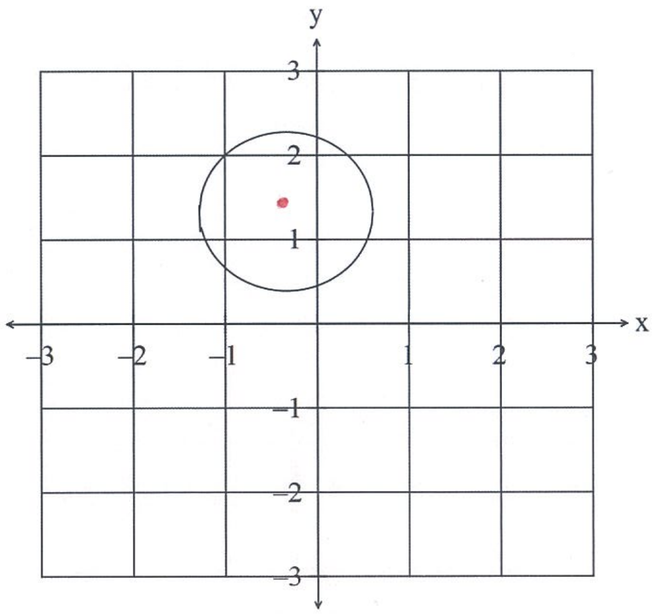
$$(x-1)^2+y^2 = 4\left(x^2+(y-1)^2\right)$$

$$x^2-2x+1+y^2 = 4\left(x^2+y^2-2y+1\right)$$

$$3x^2+3y^2+2x-8y+3=0 \quad ✓$$

$$3\left(x^2+y^2+\frac{2x}{3}-\frac{8y}{3}+1\right)=0$$

$$C\left(-\frac{1}{3},\frac{4}{3}\right) \quad r=\sqrt{\frac{1}{9}+\frac{16}{9}-1}=\sqrt{\frac{8}{9}}=\frac{2\sqrt{2}}{3} \quad ✓$$



✓

(b)

$$z = 3\frac{1+i}{1-i}\times(1+i)\times\frac{1+i}{1+i}$$

$$=3\frac{\left(1+3i+3i^2+i^3\right)}{1-i^2} \quad ✓$$

$$=\frac{3}{2}(1+3i-3-i)$$

$$z=\frac{3}{2}(-2+2i)$$

$$z=3(-1+i) \quad ✓$$

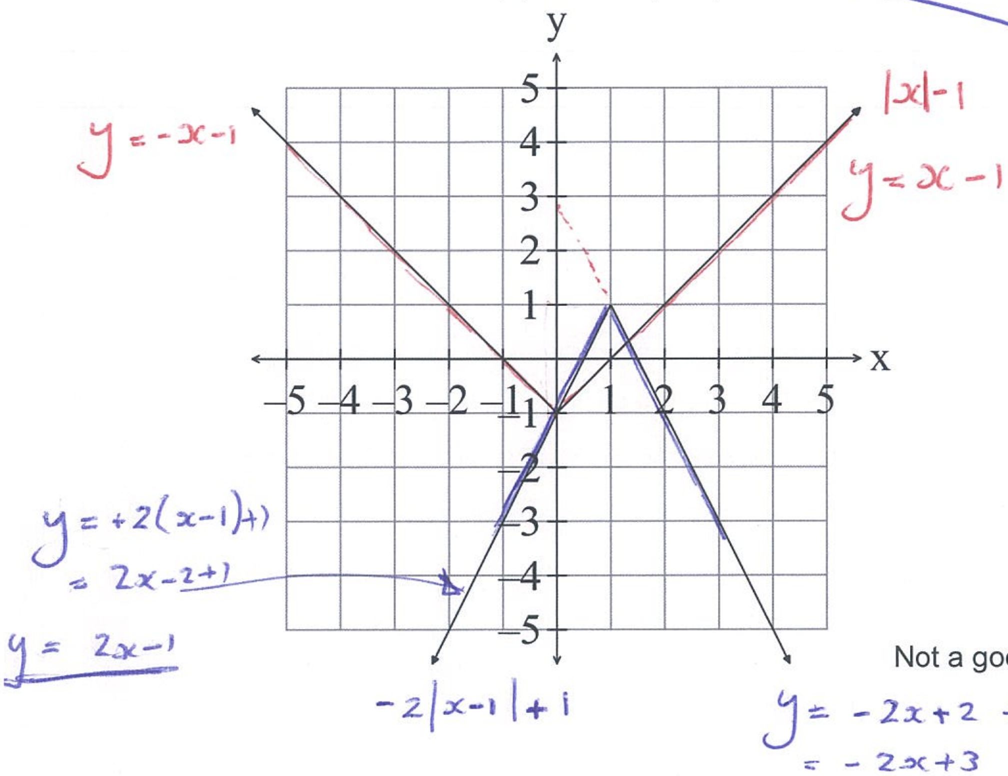
$$\therefore \ \bar{z}=3(-1-i)$$

$\bar{Z} = -3-3i$

Critical Points (Turning Pt) are @ $x=0$ or $x=1$

Question 4 **(7 marks)**

(a) Solve the equation $|x|-1=1-2|x-1|$ (5)



OR

$|x|+2|x-1|=2$

$x^2+2^2(x-1)^2=2^2$

$x^2+4(x^2-2x+1)=4$

$3x^2-4x+\cancel{4}=\cancel{4}\ 4$

$x(3x-4)=0$

$x=0$ or $x=\frac{4}{3}$

Not a good method to find fractions

$$|x|-1=\begin{cases} x-1 & for \quad x\geq 0 \\ 1-x & for \quad x<0 \end{cases}$$ ✓

$$1-2|x-1|=\begin{cases} 3-2x & for \quad x\geq 1 \\ -1+2x & for \quad x<1 \end{cases}$$ ✓

0 1

(i) $x<0$

$1-x=-1+2x$

$2=3x$

$x=\frac{2}{3}$ ✓

Not in given domain

(ii) $0\leq x<1$

$x-1=-1+2x$

$x=0$

✓

(iii) $x\geq 1$

$x-1=3-2x$

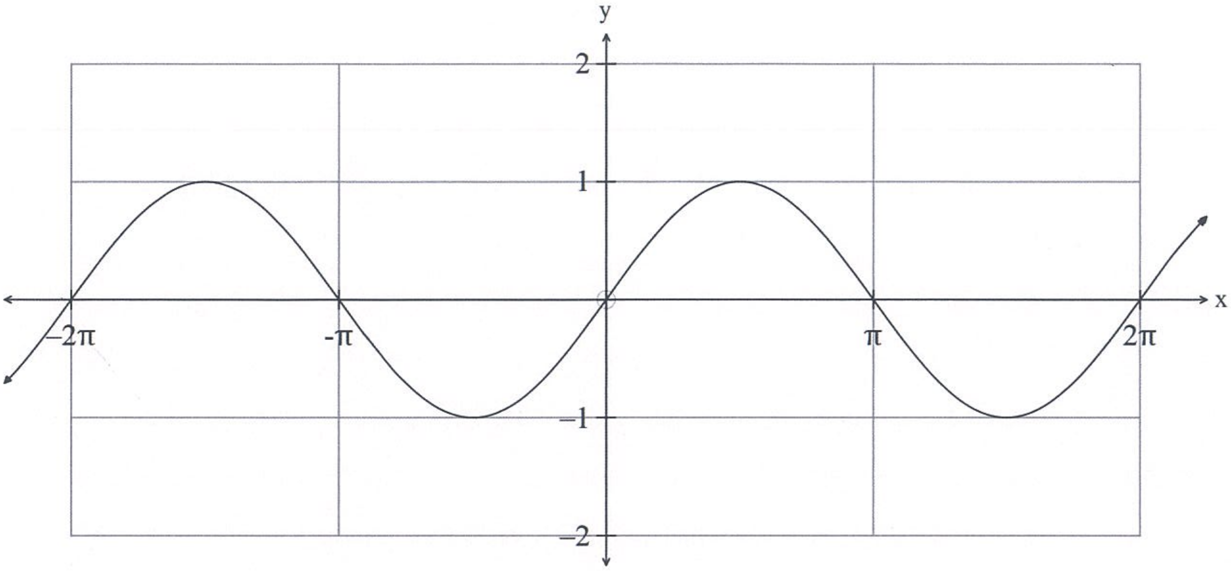
$3x=4$

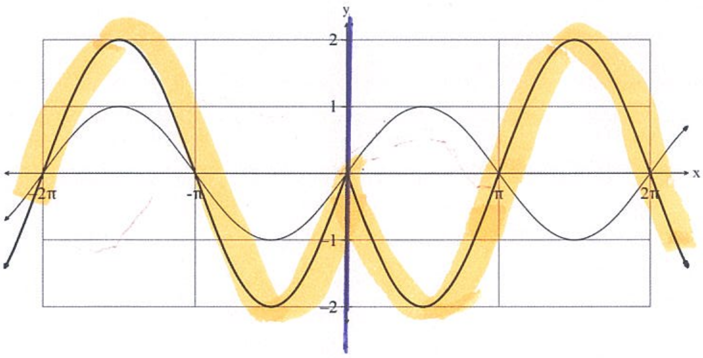
$x=\frac{4}{3}$ ✓

$x=0 \quad OR \quad x=\frac{4}{3}$

(b) The function $g(x) = sin(x)$ is sketched on the set of axes below.

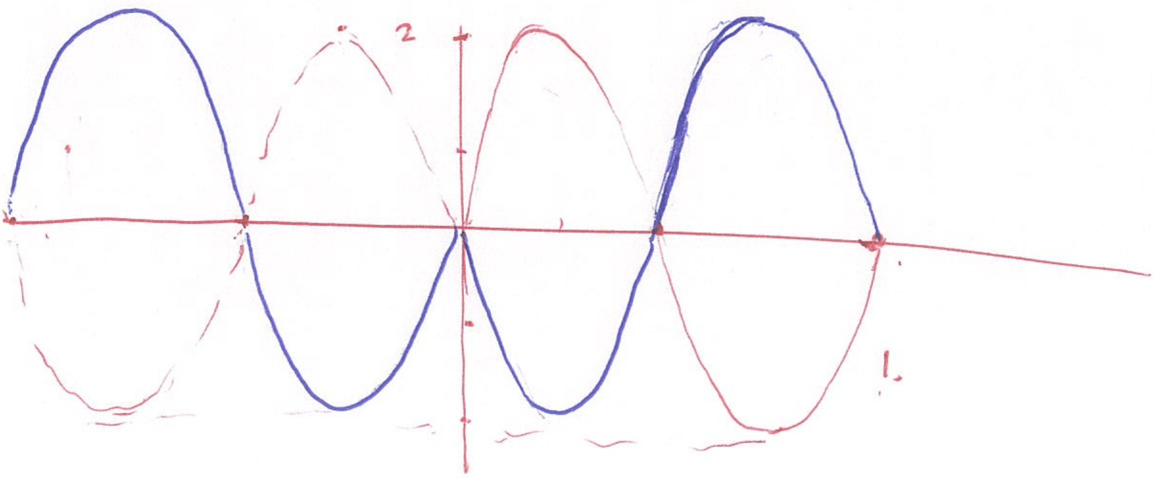
Sketch $y = -2g(|x|)$ on the set of axes below. (2)





(i) Reflect in y-axis i.e. $y = g(|x|)$

& Scale factor // y-axis ×2 $y = 2g(|x|)$



(ii) Reflect in x-axis $y = -2g(|x|)$

Question 5 **(4 marks)**

Determine $Im\left(\dfrac{(3-2i)\,cis\left(-\dfrac{5\pi}{2}\right)}{(1+i)^2}\right)$

$$= Im\left(\frac{(3-2i)(-i)}{(1+2i+i^2)}\right)$$

$$= Im\left(\frac{(-3+2i)i}{(1+2i-1)}\right)$$

$$= Im\frac{1}{2}\left(\frac{(-3+2i)i}{i}\right)$$

$$= Im\frac{1}{2}(-3+2i)$$

$$=1$$

Question 6 **(5 marks)**

The production of a chemical in a laboratory can be modelled by the differential equation $\frac{dm}{dt} = e^{2t-m}$, where m kg is the total mass of the chemical produced after t hours.

Given that $m(0) = 0$, determine an exact value for the total mass of substance produced after three hours.

$$\frac{dm}{dt} = e^{2t} \cdot e^{-m}$$

$$\int e^m \, dm = \int e^{2t} \, dt \quad ✓$$

$$e^m = \tfrac{1}{2}e^{2t} + c \quad ✓$$

$$e^0 = \tfrac{1}{2}e^0 + c \Rightarrow c = \tfrac{1}{2}$$

$$e^m = \tfrac{1}{2}e^{2t} + \tfrac{1}{2} \quad = \frac{e^{2t}+1}{2}$$

$$m = \ln\left(\tfrac{1}{2}e^{2t} + \tfrac{1}{2}\right)\Big|_{t=3} \quad ✓$$

$$= \ln\left(\frac{e^6+1}{2}\right) \quad ✓✓$$

$$= \ln(e^6+1) - \ln 2 \text{ kg} \quad ✓✓$$

full marks for Either

$$m = \ln\left(\frac{e^{2t}+1}{2}\right)$$

Question 7 **(4 marks)**

Given the vector equation of a plane is

$$\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

(a) Show that the point $(-1,7,3)$ belongs to the plane. (2)

$x = 1 - t$

$y = 2 + s + t$

$z = -4 + s + 2t$

If $x = -1,\ \ -1 = 1 - t \Rightarrow t = 2$

If $y = 7,\ \ 7 = 2 + s + 2 \Rightarrow s = 3$ ✓

If $t = 2, s = 3,\ \ z = -4 + 3 + 4 \Rightarrow z = 3$ ✓

$\therefore\ (-1,7,3) \in$ plane

The Cartesian equation of the plane is $x - y + z = -5$.

(b) Find the equation of a line that is perpendicular to the plane and contains the point $(-1,7,3)$ (2)

(ii) $\mathbf{r}(t) = \begin{pmatrix} -1 \\ 7 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ ✓ ✓ as the vector $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ is perpendicular to the plane.

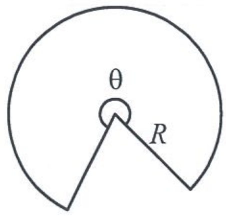
OR

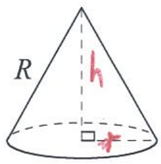
$r \cdot \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = 5$

$r \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = -5$

Question 8 **(8 marks)**

A minor sector of angle $2\pi - \theta$ is removed from a circular piece of paper of radius R. The two straight edges of the remaining major sector are pulled together to form a right circular cone, with a slant height of R.





$h = \sqrt{R^2 - r^2}$

$= \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2}$

$= \sqrt{R^2 - \frac{R^2\theta^2}{4\pi^2}}$

$= \sqrt{\frac{4\pi^2R^2 - R^2\theta^2}{4\pi^2}}$

$= \sqrt{\frac{R^2(4\pi^2 - \theta^2)}{4\pi^2}}$

$= \sqrt{\frac{R^2}{4\pi^2}} \cdot \sqrt{4\pi^2 - \theta^2}$

$\boxed{h = \frac{R}{2\pi} \cdot \sqrt{4\pi^2 - \theta^2}}$

(a) Show that the volume of the cone is given by $V = \dfrac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$. (3)

Let radius and height of cone be r and h..

Arc length of sector = circumference of cone base.

$V = \frac{1}{3}\pi r^2 \times h$

$$R\theta = 2\pi r \Rightarrow r = \frac{R\theta}{2\pi}.$$

$$h = \sqrt{R^2 - \left(\frac{R\theta}{2\pi}\right)^2} = \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2}$$

$$V = \frac{1}{3}\pi \times \left(\frac{R\theta}{2\pi}\right)^2 \times \frac{R}{2\pi}\sqrt{4\pi^2 - \theta^2}$$

$$V = \frac{R^3\theta^2\sqrt{4\pi^2 - \theta^2}}{24\pi^2}$$

(b) Find the value of θ which maximises the volume of cone. (5)

Note: R (Slant height) is a Constant

$$u = \frac{R^3\theta^2}{24\pi^2} \qquad v = \sqrt{4\pi^2 - \theta^2}$$

$$u' = \frac{R^3\theta}{12\pi^2} \qquad v' = \frac{-\theta}{\sqrt{4\pi^2 - \theta^2}}$$

$$\frac{dV}{d\theta} = \frac{R^3\theta}{12\pi^2} \times \sqrt{4\pi^2 - \theta^2} + \frac{R^3\theta^2}{24\pi^2} \times \frac{-\theta}{\sqrt{4\pi^2 - \theta^2}}$$

$$= \frac{R^3\theta}{12\pi^2}\left(\sqrt{4\pi^2 - \theta^2} - \frac{\theta^2}{2\sqrt{4\pi^2 - \theta^2}}\right)$$

i.e. $\sqrt{4\pi^2 - \theta^2} = \dfrac{\theta^2}{2\sqrt{4\pi^2 - \theta^2}}$

$= 0$ when $\theta = 0$ (Vmin) or $2(4\pi^2 - \theta^2) - \theta^2 = 0$

$\Rightarrow 8\pi^2 = 3\theta^2$

$\Rightarrow \theta = \dfrac{2\sqrt{2}}{\sqrt{3}}\pi$ (ignore -ve root as $0 < \theta < 2\pi$)

$= \sqrt{\frac{8\pi^2}{3}}$

$= \dfrac{\sqrt{8\pi^2}}{\sqrt{3}} = \dfrac{\sqrt{8}\pi}{\sqrt{3}} = \sqrt{\dfrac{8}{3}}\,\pi$

End of Questions